Kuwait University **Math 101** **Date:** **July 18, 2009**
Dept. of Math. & Comp. Sci. **First Exam** **Duration:** **90 minutes**

Calculators, cellular phones and all other mobile communication equipments are not allowed

Answer the following questions:

1. Use the definition of the limit to show that $\lim_{x \to -2} (1 - 4x) = 9$ (3 pts.)

2. Evaluate the following limits, if they exist:

   (a) $\lim_{x \to 1} \frac{\sqrt{x+8} - 3}{x^2 + 8x - 9}$ (3 pts.)

   (b) $\lim_{x \to \infty} \frac{1}{x^2} \sin(x - 1)$ (3 pts.)

3. Find the vertical and horizontal asymptotes, if any, for the graph of

$$f(x) = \frac{x\sqrt{3x^2 + 1}}{x^2 - 6x + 8}.$$ (4 pts.)

4. Find the x-coordinates of the points at which the function f is discontinuous, where

$$f(x) = \begin{cases} \dfrac{|x-2|(x-3)}{x^2 - 4} & , \text{ if } x < 3, \\ \dfrac{\tan^2(x-3)}{x-3} & , \text{ if } x > 3. \end{cases}$$

   Classify the types of discontinuity of f as removable, jump, or infinite.

   (4 pts.)

5. (a) State the Intermediate Value Theorem. (1pts.)

   (b) Let $g(x) = x^2 + f(x)$, where f is a function with continuous derivative f' (i.e., f' is continuous), such that $f'(0) = 2$ and $f'(1) = -3$. Show that $g'(x) = 0$ has a real solution.

   (3 pts.)

6. Use the definition of the derivative to find $f'(1)$, where $f(x) = \frac{1}{\sqrt{x}} + 1$.

   (4 pts.)

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1. Suppose, f is defined near a, then the statement $\lim\limits_{x\to a} f(x) = L$ means that, $\forall\epsilon > 0,\ \exists\ \delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$.
Let $\epsilon > 0$ such that $|(1-4x)-9| < \epsilon \iff |-4(x+2)| < \epsilon \iff |-4|\,|x-(-2)| < \epsilon \iff |x-(-2)| < \frac{\epsilon}{4} \iff |x-(-2)| < \delta$, where $\boxed{0 < \delta \le \frac{\epsilon}{4}}$.

2. (a) $\lim\limits_{x\to 1} \frac{\sqrt{x+8}-3}{x^2+8x-9} \times \frac{\sqrt{x+8}+3}{\sqrt{x+8}+3} = \lim\limits_{x\to 1} \frac{(x+8)-9}{(x-1)(x+9)\left[\sqrt{x+8}+3\right]} = \lim\limits_{x\to 1} \frac{1}{(x+9)\left[\sqrt{x+8}+3\right]} = \boxed{\frac{1}{60}}$

(b) <u>Solution (1):</u> Put $x = \frac{1}{y}$, as $x \to \infty$, $y \to 0^+$. So, $\lim\limits_{x\to\infty} \frac{1}{x^2}\sin(x-1) = \lim\limits_{y\to 0^+} y^2 \sin\left(\frac{1}{y}-1\right)$. Since $-1 \le \sin\left(\frac{1}{y}-1\right) \le 1$ for $y \ne 0 \implies -y^2 \le y^2\sin\left(\frac{1}{y}-1\right) \le y^2$. Since, $\lim\limits_{y\to 0^+} y^2 = 0 = \lim\limits_{y\to 0^+} -y^2$ then from the Squeeze Theorem, $\lim\limits_{y\to 0^+} y^2\sin\left(\frac{1}{y}-1\right) = 0$. Therefore, $\boxed{\lim\limits_{x\to\infty} \frac{1}{x^2}\sin(x-1) = 0}$

<u>Solution (2):</u> $-1 \le \sin(x-1) \le 1 \implies -\frac{1}{x^2} \le \frac{1}{x^2}\sin(x-1) \le \frac{1}{x^2}$. $\lim\limits_{x\to\infty}\left(-\frac{1}{x^2}\right) = 0 = \lim\limits_{x\to\infty}\left(\frac{1}{x^2}\right)$. Therefore, from the Squeeze Theorem, $\boxed{\lim\limits_{x\to\infty} \frac{1}{x^2}\sin(x-1) = 0}$

3. $f(x) = \frac{x\sqrt{3x^2+1}}{x^2-6x+8} = \frac{x\sqrt{3x^2+1}}{(x-2)(x-4)}$. $\lim\limits_{x\to 2^\pm} f(x) = \mp\infty$ & $\lim\limits_{x\to 4^\pm} f(x) = \pm\infty \implies \boxed{x = 2 \text{ and } x = 4}$ are V.A.

$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \frac{x\sqrt{x^2\left(3+\frac{1}{x^2}\right)}}{x^2\left(1-\frac{6}{x}+\frac{8}{x^2}\right)} = \lim\limits_{x\to\infty} \frac{x\sqrt{x^2}\sqrt{3+\frac{1}{x^2}}}{x^2\left(1-\frac{6}{x}+\frac{8}{x^2}\right)} = \lim\limits_{x\to\infty} \frac{x|x|\sqrt{3+\frac{1}{x^2}}}{x^2\left(1-\frac{6}{x}+\frac{8}{x^2}\right)} = \sqrt{3} \implies \boxed{y = \sqrt{3}}$ is H.A.

$\lim\limits_{x\to -\infty} f(x) = \lim\limits_{x\to -\infty} \frac{x|x|\sqrt{3+\frac{1}{x^2}}}{x^2\left(1-\frac{6}{x}+\frac{8}{x^2}\right)} = -\sqrt{3} \implies \boxed{y = -\sqrt{3}}$ is H.A.

4. $\lim\limits_{x\to -2^\pm} f(x) = \lim\limits_{x\to -2^\pm} \frac{|x-2|(x-3)}{x^2-4} = \lim\limits_{x\to -2^\pm} \frac{|x-2|(x-3)}{(x-2)(x+2)} = \boxed{\pm\infty} \implies f$ has an <u>*infinite discontinuity*</u> at $x = -2$.

$\lim\limits_{x\to 2^\pm} f(x) = \lim\limits_{x\to 2^\pm} \frac{\pm(x-2)(x-3)}{(x-2)(x+2)} = \boxed{\mp\frac{1}{4}} \implies f$ has a <u>*jump discontinuity*</u> at $x = 2$.

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} \frac{|x-2|(x-3)}{x^2-4} = 0$ & $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} \frac{\tan(x-3)\times\tan(x-3)}{x-3} = \lim\limits_{x\to 3^+} \frac{\tan(x-3)}{x-3} \times \lim\limits_{x\to 3^+} \tan(x-3) = \lim\limits_{x\to 3^+} \frac{\tan(x-3)}{x-3} \times \tan\left[\lim\limits_{x\to 3^+}(x-3)\right] = 0 \implies f$ has a <u>*removable discontinuity*</u> at $x = 3$.

5. (b) $g'(x) = 2x + f'(x)$. $g'(0) = 2(0) + f'(0) = 2 > 0$ & $g'(1) = 2(1) + f'(1) = -1 < 0$, g' is continuous on $[0,1]$. From the Intermediate Value Theorem $g'(0) = 0$ has a real solution in $(0,1)$.

6. $f'(1) = \lim\limits_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1} \frac{\left(\frac{1}{\sqrt{x}}+1\right)-2}{x-1} = \lim\limits_{x\to 1} \frac{\left(1+\sqrt{x}\right)-2\sqrt{x}}{\sqrt{x}(x-1)} = \lim\limits_{x\to 1} \frac{1-\sqrt{x}}{\sqrt{x}(x-1)} \times \frac{1+\sqrt{x}}{1+\sqrt{x}} = \lim\limits_{x\to 1} \frac{1-x}{\sqrt{x}(x-1)\left(1+\sqrt{x}\right)} = -\lim\limits_{x\to 1} \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)} = \boxed{-\frac{1}{2}}$